Math 217 Fall 2025

Quiz 10 – Solutions

Dr. Samir Donmazov

1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   > **Solution:** A function $T : V \to W$ satisfying
   >
   > $$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(\alpha v) = \alpha\, T(v)$$
   >
   > for all $u, v \in V$ and all scalars α in the underlying field (here $\mathbb{R}$). Equivalently, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all u, v and all scalars α, β.

   (b) A *subspace* of a vector space V is ...

   > **Solution:** A subset $W \subseteq V$ satisfying:
   >
   > - The zero element 0_V of V is in W.
   > - If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$.
   > - If $w \in W$, then $\lambda w \in W$ for all scalars $\lambda \in \mathbb{R}$.

   (c) Suppose U is a vector space and $u_1, u_2, \ldots, u_n \in U$. The *span* of $(u_1, u_2, \ldots, u_n)$ is ...

   > **Solution:** The set of all finite linear combinations of these vectors:
   >
   > $$\operatorname{span}\{u_1, \ldots, u_n\} = \Big\{ \sum_{i=1}^{n} \alpha_i u_i \;\Big|\; \alpha_1, \ldots, \alpha_n \in \mathbb{R} \Big\}.$$
   >
   > It is the smallest subspace of U containing all u_i.

2. Suppose V and W are vector spaces and $\vec{0}_V$ is the zero vector in V. Show: A linear transformation $T \colon V \to W$ is injective if and only if $\ker(T) = \{\vec{0}_V\}$.

> **Solution:** ($\Rightarrow$) Assume T is injective. If $v \in \ker(T)$, then $T(v) = 0_W = T(0_V)$. By injectivity, $v = 0_V$. Hence $\ker(T) = \{0_V\}$.
>
> ($\Leftarrow$) Assume $\ker(T) = \{0_V\}$. Suppose $T(u) = T(v)$. Then $T(u - v) = T(u) - T(v) = 0_W$, so $u - v \in \ker(T)$. By hypothesis $u - v = 0_V$, hence $u = v$. Therefore T is injective.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Suppose X and Y are sets. The function $f\colon X \to Y$ is injective if and only if for all $x \in X$ there is a unique $y \in Y$ such that $f(x) = y$.

   > **Solution:** FALSE. The stated property is just the *definition of a function* (well-definedness), not injectivity. For a counterexample, let
   >
   > $$X = \{1, 2\}, \quad Y = \{a\}, \quad f(1) = a, \; f(2) = a.$$
   >
   > For each $x \in X$ there is a unique $y = f(x)$, yet f is not injective since $f(1) = f(2)$ with $1 \neq 2$. Injectivity requires: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

   (b) Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. The image of T is a subspace of W.

   > **Solution:** TRUE. Let $\mathrm{Im}(T) = \{T(v) \mid v \in V\} \subseteq W$. Then $0_W = T(0_V) \in \mathrm{Im}(T)$. If $y_1 = T(v_1)$ and $y_2 = T(v_2)$, then for any scalars α, β,
   >
   > $$\alpha y_1 + \beta y_2 = \alpha T(v_1) + \beta T(v_2) = T(\alpha v_1 + \beta v_2) \in \mathrm{Im}(T).$$
   >
   > Thus $\mathrm{Im}(T)$ is closed under linear combinations, hence a subspace of W.